Draft-fin (Oct.16,2008)

NUMERICAL RANGE AND QUASI-SECTORIAL CONTRACTIONS

YURY ARLINSKIĬ AND VALENTIN ZAGREBNOV

Abstract. We apply a method developed by one of the authors, see [1], to localize the numerical range of *quasi-sectorial* contractions semigroups. Our main theorem establishes a relation between the numerical range of quasi-sectorial contraction semigroups $\{\exp(-tS)\}_{t\geq 0}$, and the maximal sectorial generators S. We also give a new prove of the rate $O(1/n)$ for the operator-norm Euler formula approximation: $\exp(-tS) = \lim_{n\to\infty}(I + tS/n)^{-n}$, $t \geq 0$, for this class of semigroups.

Contents

1. Introduction

In what follows the Banach algebra of all *bounded* linear operators on a complex Hilbert space $\mathfrak{H}$ is denoted by $\mathcal{L}(\mathfrak{H})$. We denote by $I_{\mathfrak{H}}$ the identity operator in a Hilbert space $\mathfrak{H}$. The *domain*, the *range*, and the *null-space* of a linear operator T are denoted by $\operatorname{dom} T$, $\operatorname{ran} T$, and $\ker T$, respectively. For $T \in \mathcal{L}(\mathfrak{H})$ the operators $\operatorname{Re} T = (T + T^*)/2$ and $\operatorname{Im} T = (T - T^*)/2i$ are said to be the *real* and the *imaginary* parts of T.

1.1. **Numerical range and the Kato mapping theorem.** Let $\mathfrak{H}$ be a complex separable Hilbert space and let A be an (*unbounded*) linear operator in $\mathfrak{H}$ with domain $\operatorname{dom} A$.

1991 *Mathematics Subject Classification.* 47A55, 47D03, 81Q10.
Key words and phrases. Operator numerical range; maximal sectorial generators; quasi-sectorial contractions; semigroups on the complex plane.

Definition 1.1. *The set of complex numbers*

$$W(A) := \{(Au, u) \in \mathbb{C} : \ u \in \operatorname{dom} A, \ ||u|| = 1\}$$

is called the numerical range of A, or its field of values.

According to the Hausdorff-Toeplitz theorem, the numerical range is a convex set. We recall also the following properties of the numerical range (see e.g. [14]).

Proposition 1.2. *Let A be a closed operator in $\mathfrak{H}$. Then*

(a) *for any complex number $z \notin \overline{W(A)}$ holds* $\ker (A - zI_{\mathfrak{H}}) = \{0\}$ *and* $\operatorname{ran} (A - zI_{\mathfrak{H}})$ *is closed. Moreover, the defect*

$$\operatorname{def}(A - zI_{\mathfrak{H}}) := \dim (\mathfrak{H} \ominus \operatorname{ran} (A - zI_{\mathfrak{H}}))$$

is constant in each connected component of $\mathbb{C} \setminus \overline{W(A)}$.

(b) *If $z \in \mathbb{C} \setminus \overline{W(A)}$ then*

$$||(A - zI_{\mathfrak{H}})^{-1} f|| \le \frac{1}{\operatorname{dist}(z, W(A))} \, ||f||, \ f \in \operatorname{ran} (A - zI_{\mathfrak{H}}).$$

(c) *If* $\operatorname{dom} A$ *is dense in $\mathfrak{H}$ and $W(A) \neq \mathbb{C}$, then A is closable.*

Corollary 1.3. *For a bounded operator $A \in \mathcal{L}(\mathfrak{H})$ the spectrum $\sigma(A)$ is a subset of $\overline{W(A)}$.*

For unbounded operator A the relation between spectrum and numerical range is more complicated. We would like to warn, that it may very well happen that $\sigma(A)$ is not contained in $\overline{W(A)}$, but for a closed operator A the essential spectrum $\sigma_{ess}(A)$ is always a subset of $\overline{W(A)}$. The condition $\operatorname{def}(A - zI) = 0, \ z \notin \overline{W(A)}$ in Proposition 1.2 serves to ensure that for those unbounded operators one gets

$$\sigma(T) \subset \overline{W(A)} \,, \tag{1.1}$$

i.e., the same conclusion as in Corollary 1.3 for bounded operators.

In the sequel we need the following numerical range mapping theorem due to Kato [13].

Proposition 1.4. [13]. *Let $f(z)$ be a rational function on $\mathbb{C}$, with $f(\infty) = \infty$. Let for some compact and convex set $E' \subset \mathbb{C}$ the inverse function $f^{-1} : E' \to E \supseteq K$, where K is a convex kernel of E, i.e., is a subset of E such that E is star-shaped with respect to any $z \in K$. If A is bounded operator with $W(A) \subseteq K$, then $W(f(A)) \subseteq E'$.*

Notice that for a convex set E the *maximal* convex kernel $K = E$.

1.2. **Sectorial operators and quasi-sectorial contractions.**

Definition 1.5. [14]. *Let S be a linear operator in a Hilbert space $\mathfrak{H}$. If* $\operatorname{Re}(Su, u) \ge 0$ *for all $u \in \operatorname{dom} S$, then S is called accretive.*

So, the operator S is accretive if and only if its numerical range is contained in the closed right-half plane of the complex plane. An accretive operator S is called *maximal* accretive (m-accretive) if one of the equivalent conditions is satisfied:

- the operator S has no accretive extensions in $\mathfrak{H}$;
- the resolvent set $\rho(S)$ is nonempty;
- the operator S is densely defined and closed, and S^* is accretive operator.

The resolvent set $\rho(S)$ of m-accretive operator contains the open left half plane and

$$||(S - zI_{\mathfrak{H}})^{-1}|| \le \frac{1}{|\operatorname{Re} z|} \,, \ \operatorname{Re} z < 0.$$

It is well known [14] that if S is m-accretive operator, then the one-parameter semigroup

$$T(t) = \exp(-tS),\ t \geq 0$$

is contractive. Conversely, if the family $\{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup of bounded operators in a Hilbert space $\mathfrak{H}$, with $T(0) = I_{\mathfrak{H}}$ (C_0-semigroup) and $T(t)$ is a contraction for each t, then the generator S of $T(t)$:

$$Su := \lim_{t\to+0} \frac{(I_{\mathfrak{H}} - T(t))u}{t}\ ,\ u \in \operatorname{dom} S\ ,$$

where domain is defined by condition:

$$\operatorname{dom} S = \left\{u \in \mathfrak{H} : \lim_{t\to+0} \frac{(I_{\mathfrak{H}} - T(t))u}{t} \quad \text{exists}\right\}\ ,$$

is an m-accretive operator in $\mathfrak{H}$. Then the Euler formula approximation:

$$T(t) = s - \lim_{n\to\infty} \left(I_{\mathfrak{H}} + \frac{t}{n}S\right)^{-n}\ ,\ t \geq 0 \tag{1.2}$$

holds in the strong operator topology, see e.g. [14].

Definition 1.6. [14]. *Let* $\alpha \in [0, \pi/2)$ *and let*

$$\mathcal{S}(\alpha) := \{z \in \mathbb{C} : |\arg z| \leq \alpha\}$$

be a sector on the complex plane $\mathbb{C}$ *with the vertex at the origin and the semi-angle* α.

A linear operator S *in a Hilbert space* $\mathfrak{H}$ *is called sectorial with vertex at* $z = 0$ *and the semi-angle* α *if* $W(S) \subseteq \mathcal{S}(\alpha)$.

If S is m-accretive and sectorial with vertex at $z = 0$ and the semi-angle α then it is called m-sectorial with vertex at the origin and with the semi-angle α. For short we call these operators m-α-sectorial. The resolvent set of m-α-sectorial operator S contains the set $\mathbb{C} \setminus \mathcal{S}(\alpha)$ and

$$||(S - zI_{\mathfrak{H}})^{-1}|| \leq \frac{1}{\operatorname{dist}(z, \mathcal{S}(\alpha))}\ ,\ z \in \mathbb{C} \setminus \mathcal{S}(\alpha).$$

It is well-known [14] that

- a C_0-semigroup $T(t) = \exp(-tS)$, $t \geq 0$ has *contractive* and *holomorphic* continuation into the sector $\mathcal{S}(\pi/2 - \alpha)$ if and only if the generator S is m-α-sectorial operator,
- the *sesquilinear* form: $(Su, v)\ ,\ u, v \in \operatorname{dom} S$ is closable.

Denote by $\mathfrak{S}[\cdot,\cdot]$ the closure of the form (Su, v) and by $\mathcal{D}[\mathfrak{S}]$ its domain. By the *first representation theorem* [14] the operator S is *uniquely* associated with the form $\mathfrak{S}[\cdot,\cdot]$ in the following sense:

$$(Su, v) = \mathfrak{S}[u, v] \quad \text{for all} \quad u \in \operatorname{dom} S \quad \text{and for all} \quad v \in \mathcal{D}[\mathfrak{S}].$$

The important relation between the one-parameter semigroups $\{T(t) := \exp(-tS)\}_{t\geq 0}$ generated by the m-α-sectorial operators S and the corresponding closed sesquilinear forms is established in [3]:

$$u \in \mathcal{D}[\mathfrak{S}] = \left\{u \in \mathfrak{H} : \frac{d}{dt}(T(t)u, u)\Big|_{t=+0} \text{ exits}\right\},$$

and one has that

$$\frac{d}{dt}(T(t)u, u)\Big|_{t=+0} = -\mathfrak{S}[u, u]\ ,\ u \in \mathcal{D}[\mathfrak{S}]\ . \tag{1.3}$$

Notice that in this case of the Euler approximation (1.2) converges to the semigroup in the *operator-norm* topology [7], [8], [19].

Definition 1.7. [8]. *For any $\alpha \in [0, \pi/2)$ we define in the complex plane $\mathbb{C}$ a closed domain:*

$$D_\alpha := \{z \in \mathbb{C} : |z| \leq \sin\alpha\} \cup \{z \in \mathbb{C} : |\arg(1-z)| \leq \alpha \text{ and } |z-1| \leq \cos\alpha\} \ . \tag{1.4}$$

This is a convex subset of the unit disc $\mathbb{D} = D_{\alpha=\pi/2}$, with the "angle" (in contrast to the "tangent") touching of the boundary $\partial\mathbb{D}$ at the only one point $z = 1$. It is evident that $D_\alpha \subset D_{\beta>\alpha}$.

Definition 1.8. [8]. *A contraction C on the Hilbert space $\mathfrak{H}$ is called quasi-sectorial with semi-angle $\alpha \in [0, \pi/2)$, if its numerical range $W(C) \subseteq D_\alpha$.*

It is evident that if operator C is a *quasi-sectorial* contraction, then $I - C$ is an *m-sectorial* operator with vertex $z = 0$ and semi-angle α. The limits $\alpha = 0$ and $\alpha = \pi/2$ correspond, respectively, to non-negative (i.e. *self-adjoint*) and to some *general* contraction.

Remark 1.9. [8]. *Notice that the resolvent family $\{(I_\mathfrak{H} + t\,S)^{-1}\}_{t\geq 0}$ of the m-α-sectorial operator S, gives the first non-trivial example of a quasi-sectorial contractions, if one considers the semi-angles $\alpha \in [0, \pi/3)$. Below (see Section 2) we show that it can be extended to $\alpha \in [0, \pi/2)$.*

The definition of quasi-sectorial contractions was motivated in [8] by the lifting of the Trotter-Kato product formula and the Chernoff theory of semigroup approximation [12] to the operator-norm topology.

Namely, using properties of quasi-sectorial contractions established in [8] for the case of m-α-sectorial generator, it was proved that there is *operator-norm* convergence of the Euler semigroup approximation (1.2) and of the Trotter product formula, see [7], [8], [16], [6], [19]. Theorem 2.1 from [8] states that the operators $T(t) = \exp(-tS)$ are quasi-sectorial contractions with $W(T(t)) \subset D_\alpha$ for all $t \geq 0$ and $\alpha \in [0, \pi/2)$. Here operator S stands for m-α-sectorial generator. As it is indicated in [18], the proof in [8] has a flaw. In [18] it was corrected, but only for the semi-angles $\alpha \in [0, \pi/4]$.

Our main Theorem 3.4 establishes a quite accurate relation between m-α-sectorial generators and the numerical range of the corresponding one-parameter contraction semigroups. It improves the recent result [18] from $\alpha \in [0, \pi/4]$ to $\alpha \in [0, \pi/2)$. To this end we use in the next section some nontrivial results due to [1]-[4], concerning a class of operator contractions and of semigroups on the complex plane. Besides that we use recent results related to the generalizations of the famous von Neumann inequality [15], obtained in [5], [9], [10], [11], in order to give a new prove that in fact the Euler formula (1.2) converges in the operator-norm with the rate $O(1/n)$.

2. Operators of the class $C_\mathfrak{H}(\alpha)$

A fundamental for us will be the class of contractions introduced for the first time in [1] and studied in [1]-[2]:

Definition 2.1. [1]. *Let $\alpha \in (0, \pi/2)$. We say that a bounded operator $T \in \mathcal{L}(\mathfrak{H})$ belongs to the class $C_\mathfrak{H}(\alpha)$ if*

$$||T\,\sin\alpha \pm iI_\mathfrak{H}\cos\alpha|| \leq 1. \tag{2.1}$$

It is clear, that the class $C_\mathfrak{H}(\alpha)$ is a convex and closed (with respect to the strong operator topology) set, which is intersection of two closed operator balls corresponding to $\pm$. Moreover, by virtue of (2.1) one immediately concludes that:

$$T \in C_\mathfrak{H}(\alpha) \iff -T \in C_\mathfrak{H}(\alpha) \iff T^* \in C_\mathfrak{H}(\alpha),$$

and that condition (2.1) is equivalent to the following criterium:

$$\tan\alpha\,(||f||^2 - ||Tf||^2) \geq 2|\mathrm{Im}\,(Tf,f)|,\ f \in \mathfrak{H}. \tag{2.2}$$

This inequality implies that the operator T is a contraction. Together with Definition 1.1, it also proves that $T \in C_{\mathfrak{H}}(\alpha)$ is equivalent to the statement that $(I - T^*)(I + T)$ is a bounded m-α-sectorial operator.

According to (2.2) it is natural to identify $C_{\mathfrak{H}}(\alpha = 0)$ with the set of all *self-adjoint* contractions whereas $C_{\mathfrak{H}}(\alpha = \pi/2)$ is the set of general contractions on $\mathfrak{H}$.

Now following [4] we define for $\alpha \in (0, \pi/2)$ the family of subsets of the complex plane $\mathbb{C}$:

$$C(\alpha) = \{z \in \mathbb{C} : |z\sin\alpha \pm i\cos\alpha| \leq 1\} = \left\{z \in \mathbb{C} : (1 - |z|^2)\tan\alpha \geq 2|\mathrm{Im}\,z|\right\}, \tag{2.3}$$

cf. definitions (2.1), (2.2). Then, similar to (2.1), each set $C(\alpha)$ is the intersection of *two* closed disks of the complex plane which is contained in the closed unit disk $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. From (2.3) it is clear that $C(\alpha = 0) = [-1, 1]$.

Remark 2.2. *By virtue of definitions (2.1), (2.2) and (2.3) one gets that $T \in C_{\mathfrak{H}}(\alpha)$ implies for the numerical range: $W(T) \subseteq C(\alpha)$. Then one has:*

$$||\mathrm{Im}\,T|| \leq \tan(\alpha/2)\ . \tag{2.4}$$

Notice that $C(\alpha) \setminus D_\alpha \neq \emptyset$, i.e. in general the operator T is not quasi-sectorial.

Besides, the operator class $C_{\mathfrak{H}}(\alpha)$ has several interesting properties. In particular the following one [1]:

Proposition 2.3. [1]. *Let T_1 and T_2 belong to the class $C_{\mathfrak{H}}(\alpha)$. Then the operator*

$$(T_1T_2 + T_2T_1)/2$$

belongs to the class $C_{\mathfrak{H}}(\alpha)$. In particular, if T_1 and T_2 are two commuting operators from the class $C_{\mathfrak{H}}(\alpha)$, then the product $T = T_1\,T_2$ also belongs to $C_{\mathfrak{H}}(\alpha)$.

Remark 2.4. *Notice that the set $C(\alpha)$ inherits a similar property: for each $\alpha \in (0, \pi/2)$ it forms a multiplicative semigroup of complex numbers in the plane $\mathbb{C}$. Detailed properties of these semigroups have been studied in* [4].

Now we are in position to establish a connection between set of contractions $C_{\mathfrak{H}}(\alpha)$ and m-α-sectorial operators.

Proposition 2.5. [1]. *If S is m-α-sectorial operator, then*

$$T = (I_{\mathfrak{H}} - S)(I_{\mathfrak{H}} + S)^{-1}$$

belongs to the class $C_{\mathfrak{H}}(\alpha)$ and conversely, if $T \in C_{\mathfrak{H}}(\alpha)$, and $\ker(I_{\mathfrak{H}} + T) = \{0\}$, then $S = (I_{\mathfrak{H}} - T)(I_{\mathfrak{H}} + T)^{-1}$ is m-α-sectorial operator.

Now let S be m-α-sectorial operator and let $\lambda > 0$. Then λS is also m-α-sectorial. By Proposition 2.5 the operator $U(\lambda) = (I_{\mathfrak{H}} - \lambda S)(I_{\mathfrak{H}} + \lambda S)^{-1}$ belongs to the class $C_{\mathfrak{H}}(\alpha)$. Let us put

$$F(\lambda) := (I_{\mathfrak{H}} + \lambda S)^{-1} = \frac{1}{2}(U(\lambda) + I_{\mathfrak{H}}). \tag{2.5}$$

Since the operators $U(\lambda)$ and $I_{\mathfrak{H}}$ belong to the class $C_{\mathfrak{H}}(\alpha)$ and since it is a convex set, the operator $F(\lambda)$ is also in the class $C_{\mathfrak{H}}(\alpha)$. Hence, by Remark 2.2 one obtains $W(F(\lambda)) \subseteq C(\alpha)$.

Remark 2.6. *Following the arguments of* [8],[18] *we can localize the numerical range* $W(F(\lambda))$ *even more accurate, cf. Remark 1.9. Since for any* $u \in \mathfrak{H}$ *we have:*

$$(u, F(\lambda)u) = (v_\lambda, v_\lambda) + \lambda(Sv_\lambda, v_\lambda) \in \mathcal{S}(\alpha) \ ,$$

where $v_\lambda := F(\lambda)u$*, it follows that*

$$W(F(\lambda)) \subseteq \mathcal{S}(\alpha) \cap C(\alpha) \ ,$$

for $\lambda > 0$ *and* $\alpha \in [0, \pi/2)$*. Moreover, since* $U(\lambda) = 2F(\lambda) - I_{\mathfrak{H}} \in C_{\mathfrak{H}}(\alpha)$*, see (2.5), we find that*

$$\|(F(\lambda) - I_{\mathfrak{H}}/2)\sin\alpha \pm iI_{\mathfrak{H}}(\cos\alpha)/2\| \leq 1/2 \ . \tag{2.6}$$

For the numerical range this implies:

$$W(F(\lambda)) \subseteq L(\alpha) := \{\zeta \in \mathbb{C} : |(\zeta - 1/2)\sin\alpha \pm (i\cos\alpha)/2| \leq 1/2\} \ , \quad \text{for} \quad \lambda > 0 \ . \tag{2.7}$$

Notice that $L(\alpha) \subset \mathcal{S}(\alpha) \cap C(\alpha)$*, see Figure 1.*

Now, we follow essentially the line of reasoning of [1] to establish the one-to-one correspondence between m-α-sectorial generators and contraction semigroups of the class $C_{\mathfrak{H}}(\alpha)$, cf. Remark 2.2 about relation to quasi-sectorial contractions.

Theorem 2.7. *If* S *is* m-α*-sectorial operator in a Hilbert space* $\mathfrak{H}$*, then the corresponding semigroup* $T(t) = \exp(-tS) \in C_{\mathfrak{H}}(\alpha)$*, for all* $t \geq 0$*. Conversely, let* $T(t) = \exp(-tS)$ *for* $t \geq 0$ *be a* C_0*-semigroup of contractions on a Hilbert space* $\mathfrak{H}$*. If* $T(t) \in C_{\mathfrak{H}}(\alpha)$ *for non-negatives* t *in neighborhood of* $t = 0$*, then the generator* S *is an* m-α*-sectorial operator.*

Proof. Let S be a m-α-sectorial operator and let $\lambda \geq 0$. By (2.5) the operator $F(\lambda) = (I_{\mathfrak{H}} + \lambda S)^{-1}$ belongs to the class $C_{\mathfrak{H}}(\alpha)$. Therefore, by Proposition 2.3 for each $t \geq 0$ and any natural number n the operator

$$F^n\left(\frac{t}{n}\right) = \left(I_{\mathfrak{H}} + \frac{t}{n}S\right)^{-n}$$

belongs to the class $C_{\mathfrak{H}}(\alpha)$. Taking in account that the set $C_{\mathfrak{H}}(\alpha)$ is closed with respect to the strong operator topology, from the Euler formula (1.2) we get that strong limit $T(t) = \exp(-tS)$ also belongs to the class $C_{\mathfrak{H}}(\alpha)$.

Now suppose that semigroup $T(t) = \exp(-tS) \in C_{\mathfrak{H}}(\alpha)$ for $t \in [0, \delta)$, where $\delta > 0$. Define operator family:

$$B_{\pm}(t) := T(t)\,\sin\alpha \pm i\cos\alpha I_{\mathfrak{H}},\ t \geq 0.$$

Since $B_{\pm}(0) = (\sin\alpha \pm i\cos\alpha)I_{\mathfrak{H}}$ and $T(t) \in C_{\mathfrak{H}}(\alpha)$ for $t \in [0, \delta)$, we get

$$||B_{\pm}(t)f||^2 \leq ||f||^2 = ||B_{\pm}(0)||f||^2,\ t \in [0, \delta),\ f \in \mathfrak{H}.$$

Since

$$||B_{\pm}(t)f||^2 = \sin^2\alpha||T(t)f||^2 + \cos^2\alpha||f||^2 \pm 2\sin\alpha\,\cos\alpha\,\mathrm{Im}\,(T(t)f, f),\ f \in \mathfrak{H},$$

for all $f \in \mathrm{dom}\, S$ we have:

$$\frac{d}{dt}\left(||B_{\pm}(t)f||^2\right)\Big|_{t=+0} = -2\sin^2\alpha\,\mathrm{Re}\,(Sf, f) \pm 2\sin\alpha\,\cos\alpha\,\mathrm{Im}\,(Sf, f) \leq 0.$$

Thus, $W(S) \subseteq \mathcal{S}(\alpha)$. But since operator S is m-accretive, it is m-α-sectorial [14]. $\square$

3. Numerical range for contractive holomorphic semigroups

From Theorem 2.7 it follows, in particular, that for m-α-sectorial generator S the numerical range of the corresponding contraction semigroup

$$W(\exp(-tS)) \subseteq C(\alpha) \quad \text{for all} \quad t \geq 0\ .$$

But as we warranted in Remark 2.2 it does not imply that this semigroup is quasi-sectorial contraction. It was discovered in [8] that the conformal mapping : $z \mapsto z^2$, together with the Kato numerical range theorem (Proposition 1.4) play a special rôle in the theory of quasi-sectorial contractions.

Definition 3.1. *Let $\alpha \in [0, \pi/2)$. We define a domain:*

$$\Omega(\alpha) := \left\{z^2 : z \in C(\alpha)\right\}. \tag{3.1}$$

So, if $f(z) = z^2$, then $\Omega(\alpha) = f(C(\alpha))$. Since (see Remark 2.4) $C(\alpha)$ is a *multiplicative semigroup*, we obtain that $\Omega(\alpha) \subseteq C(\alpha)$, and that the subset $\Omega(\alpha)$ is in turn a multiplicative semigroup [4].

From (2.3) and Proposition 2.3 it follows then that for any $\alpha \in (0, \pi/2)$ the set (3.1) has representation:

$$\Omega(\alpha) = \left\{z \in \mathbb{C} : |\sqrt{z}\sin\alpha \pm i\cos\alpha| \leq 1\right\} = \left\{z \in \mathbb{C} : 2|\mathrm{Im}\,\sqrt{z}| \leq (1 - |z|)\tan\alpha\right\}, \tag{3.2}$$

with the limiting cases: $\Omega(\alpha = 0) = [0, 1]$ and $\Omega(\alpha = \pi/2) = \overline{\mathbb{D}}$.

Lemma 3.2. *The set $\Omega(\alpha)$ (3.1) is convex and $\Omega(\alpha) \subseteq D_\alpha$.*

Proof. Let $C_+(\alpha) := \{z \in C(\alpha) : \mathrm{Im}\, z \geq 0\}$. Then clearly, $\Omega(\alpha) = f(C_+(\alpha))$, where $f(z) = z^2$. Denote

$$\Gamma(\alpha) = \partial C_+(\alpha) \setminus (-1, 1).$$

Then $\partial\Omega(\alpha) = f(\Gamma(\alpha))$. Since

$$\Gamma(\alpha) = \left\{z : z = \frac{e^{it} - i\cos\alpha}{\sin\alpha},\ t \in \left[\frac{\pi}{2} - \alpha, \frac{\pi}{2} + \alpha\right]\right\},$$

the boundary $\partial\Omega(\alpha)$ can be parameterized as follows:

$$\partial\Omega(\alpha) = \left\{z = \zeta(t) = \frac{(e^{it} - i\cos\alpha)^2}{\sin^2\alpha},\ t \in \left[\frac{\pi}{2} - \alpha, \frac{\pi}{2} + \alpha\right]\right\}. \tag{3.3}$$

Put $x = x(t) := \mathrm{Re}\,\zeta(t)$, $y = y(t) := \mathrm{Im}\,\zeta(t)$. Then, since

$$\frac{d^2y}{dx^2} = \frac{y''(t)x'(t) - y'(t)x''(t)}{(x'(t))^3},$$

we obtain

$$\frac{d^2y}{dx^2} = -\frac{\mathrm{Im}\,(\zeta'(t)\overline{\zeta''(t)})}{(\mathrm{Re}\,\zeta'(t))^3}.$$

Further, by explicit calculations we get:

$$\zeta'(t) = \frac{2ie^{it}(e^{it} - i\cos\alpha)}{\sin^2\alpha}\ ,\quad \zeta''(t) = \frac{2ie^{it}(2ie^{it} + \cos\alpha)}{\sin^2\alpha},$$

$$-\mathrm{Im}\,(\zeta'(t)\overline{\zeta''(t)}) = \frac{4(2 + \cos^2\alpha - 3\cos\alpha\sin t)}{\sin^4\alpha}\ ,\quad \mathrm{Re}\,\zeta'(t) = \frac{2\cos t(\cos\alpha - 2\sin t)}{\sin^2\alpha}.$$

For $t \in [\pi/2 - \alpha, \pi/2 + \alpha]$, by the estimates

$$\cos^2\alpha - 3\cos\alpha\sin t + 2 \geq 2\cos^2 - 3\cos\alpha + 2 = (2 - \cos\alpha)(1 - \cos\alpha) > 0,$$

we obtain $-\mathrm{Im}\,(\zeta'(t)\overline{\zeta''(t)}) > 0$.

Since $\cos\alpha - 2\sin t \le -\cos\alpha < 0$, then for $t \in [\pi/2-\alpha, \pi/2+\alpha]$, we get that

$$\frac{d^2y}{dx^2} < 0 \quad \text{for} \quad t \in \left[\frac{\pi}{2} - \alpha, \frac{\pi}{2}\right) ,$$
$$\frac{d^2y}{dx^2} > 0 \quad \text{for} \quad t \in \left(\frac{\pi}{2}, \frac{\pi}{2} + \alpha\right] ,$$

which implies that the set $\Omega(\alpha)$ is convex.

Since the mapping: $z \mapsto z^2$ is conformal and $\Omega(\alpha)$ is convex, the proof of the second part of the lemma follows from the estimate

$$-(\tan\alpha/2)^2 \le \operatorname{Re}\zeta(t) , \tag{3.4}$$

where $(\tan\alpha/2)^2 \le \sin\alpha$ for $\alpha \in (0, \pi/2)$, see (1.4) and (3.3). □

Notice that in view of relation:

$$y(t) = \frac{\sin 2t - 2\cos\alpha\cos t}{\sin^2\alpha} ,$$

one obtains

$$\max_{z\in\Omega(\alpha)} |\operatorname{Im} z| = \frac{\sin 2\gamma - 2\cos\alpha\sin\gamma}{\sin^2\alpha} , \tag{3.5}$$

where

$$\sin\gamma = \frac{\cos\alpha + \sqrt{\cos^2\alpha + 8}}{4} , \quad \gamma \in (0, \pi/2) . \tag{3.6}$$

Since $\Omega(\alpha) \subset C(\alpha)$, the number in the right-hand side of (3.5) is less than $\tan(\alpha/2)$.

Lemma 3.3. *Let $\alpha \in [0, \pi/2)$ and let $T \in C_{\mathfrak{H}}(\alpha)$. Then $W(T^{2n}) \subseteq \Omega(\alpha)$ for all natural numbers n.*

Proof. We apply the Kato numerical range mapping theorem (Proposition 1.4) for $f(z) = z^2$ with $E' = \Omega(\alpha)$. Then by Definition 3.1 we have $E := f^{-1}(\Omega(\alpha)) = C(\alpha)$. Since $C(\alpha)$ is a convex set, its maximal convex kernel K *coincides* with $C(\alpha)$. By virtue of Proposition 2.3 from $T \in C_{\mathfrak{H}}(\alpha)$ it follows that for all natural n one gets $T^n \in C_{\mathfrak{H}}(\alpha)$. Then by Remark 2.2 we obtain $W(T^n) \subseteq C(\alpha) = K$. Applying now the Kato mapping theorem for $f(z) = z^2$ we obtain $W(T^{2n}) \subseteq \Omega(\alpha)$. □

Now we are in position to proof the main theorem of the present paper.

Theorem 3.4. (1) *Let S be m-α-sectorial operator. Then*

$$W(\exp(-tS)) \subseteq \Omega(\alpha) , \ t \ge 0. \tag{3.7}$$

In particular, $\{\exp(-tS)\}_{t\ge0}$ is the quasi-sectorial contraction semigroup.

(2) *The inverse is also true. Let $\{T(t) := \exp(-tS)\}_{t\ge0}$ be a C_0-semigroup on a Hilbert space $\mathfrak{H}$. If in some neighborhood of $t = 0$ the numerical range: $W(T(t \ge 0)) \subseteq \Omega(\alpha)$ for some $\alpha \in [0, \pi/2)$, then the generator S is an m-α-sectorial operator.*

Proof. (1) By Theorem 2.7 the operators $T(t) = \exp(-tS)$ belong to the class $C_{\mathfrak{H}}(\alpha)$ for all $t \ge 0$, but $T(t) = T^2(t/2)$, then by Lemma 3.3 we obtain $W(T(t)) \subseteq \Omega(\alpha)$. Since Lemma 3.2 implies $\Omega(\alpha) \subset D_\alpha$, the operators $T(t)$ are quasi-sectorial contractions.

(2) Fix $u \in \operatorname{dom} S$, $||u|| = 1$. For $t \in [0, \delta]$ we define two functions:

$$G_\pm(t) := |\sin\alpha\sqrt{(T(t)u, u)} \pm i\cos\alpha|^2 .$$

Then $G_{\pm}(0) = 1$. The condition $W(T(t)) \subseteq \Omega(\alpha)$ yields (see (3.2)) that $G_{\pm}(t) \leq G_{\pm}(0)$, $t \in [0, \delta]$, which implies:

$$\frac{d}{dt}\left(G_{\pm}(t)\right)\Big|_{t=+0} \leq 0 \ .$$

Therefore we get that

$$\lim_{t\to+0} \frac{1 - G_{\pm}(t)}{t} \geq 0. \tag{3.8}$$

Further we use the following identities:

$$\begin{aligned}&1 - G_{\pm}(t) = \sin^2\alpha(1 - |(T(t)u,u)|) \mp 2\sin\alpha\,\cos\alpha\,\mathrm{Im}\sqrt{(T(t)u,u)} = \\ &= \frac{\sin^2\alpha\left(1 - (T(t)u,u) + (T(t)u,u)(1 - \overline{(T(t)u,u)})\right)}{1 + |(T(t)u,u|} \mp 2\sin\alpha\,\cos\alpha\,\mathrm{Im}\sqrt{(T(t)u,u)},\end{aligned}$$

$$\begin{aligned}\frac{1 - G_{\pm}(t)}{t} &= \frac{\sin^2\alpha\left(1 - (T(t)u,u) + (T(t)u,u)(1 - \overline{(T(t)u,u)})\right)}{t(1 + |(T(t)u,u|)} \mp \\ &\mp 2\sin\alpha\,\cos\alpha\,\mathrm{Im}\left(\frac{(T(t)u,u) - 1}{t(\sqrt{(T(t)u,u)} + 1)}\right) \ .\end{aligned}$$

Since by (1.3) one has:

$$\lim_{t\to+0} \frac{1 - (T(t)u,u)}{t} = (Su,u) \ ,$$

and by (3.8) we get

$$\lim_{t\to+0} \frac{1 - G_{\pm}(t)}{t} = \sin^2\alpha\,\mathrm{Re}\,(Su,u) \pm \sin\alpha\,\cos\alpha\,\mathrm{Im}\,(Su,u) \geq 0 \ .$$

The last estimate implies that $W(S) \subseteq \mathcal{S}(\alpha)$, i.e., S is m-α-sectorial operator. □

Thus, the two equivalent conditions: $\exp(-tS) \in C_{\mathfrak{H}}(\alpha)$ (see Theorem 2.7) and $W(\exp(-tS)) \subseteq \Omega(\alpha)$ *completely* characterize m-α-sectorial operators S. Since (3.7) yields $W(\exp(-tS)) \subset D_\alpha$, the statement in Theorem 3.4(1) is extension of Theorem 2.1 (proven in [18] for $\alpha \in [0, \pi/4)$) to the whole class of the m-α-sectorial generators: $\alpha \in [0, \pi/2)$.

Notice that taking into account the multiplicative semigroup property of the set $\Omega(\alpha)$ and Theorem 3.4 we get by consequence the following inequalities:

$$\left|\sin\alpha\left(\prod_{k=1}^{n}(\exp(-t_kS_k)u_k, u_k)\right)^{1/2} \pm i\cos\alpha\right| \leq 1$$

for arbitrary m-α-sectorial operators $S_1, \dots S_n$, any non-negative numbers $t_1, \dots, t_n$, and arbitrary normalized vectors $u_1, \dots, u_n$ from $\mathfrak{H}$. By semi-group property:
$\exp(-t_1S)\exp(-t_2S) = \exp(-(t_1 + t_2)S)$. Hence we also get

$$|\sin\alpha\sqrt{(\exp(-t_1S)u, \exp(-t_2S^*)u)} \pm i\cos\alpha| \leq 1$$

for any $t_1, t_2 \geq 0$ and any $u \in \mathfrak{H}$, $||u|| = 1$.

Remark that since $\mathrm{Re}\,z \geq -\tan^2(\alpha/2)$ for all $z \in \Omega(\alpha)$ and $W(\exp(-tS)) \subseteq \Omega(\alpha)$, we obtain:

$$\mathrm{Re}\,\exp(-tS) \geq -\tan^2(\alpha/2)\ I_{\mathfrak{H}}\ ,\ t \geq 0\ .$$

The inclusion $W(\exp(-tS)) \subseteq \Omega(\alpha)$ also implies (see (3.5)) that

$$||\mathrm{Im}\,\exp(-tS)|| \leq \frac{\sin 2\gamma - 2\cos\alpha\sin\gamma}{\sin^2\alpha} < \tan(\alpha/2)\ ,\ t \geq 0\ ,$$

where γ satisfies (3.6). By virtue of relation $\mathrm{Re}\,T^2 = (\mathrm{Re}\,T)^2 - (\mathrm{Im}\,T)^2$ (valid for any bounded operator T) and since one has that $\mathrm{Re}\,\exp(-tS) = (\mathrm{Re}\,\exp(-tS/2))^2 - (\mathrm{Im}\,\exp(-tS/2))^2$, we obtain:

$$\mathrm{Re}\,\exp(-tS) \geq -\left(\frac{\sin 2\gamma - 2\cos\alpha\sin\gamma}{\sin^2\alpha}\right)^2 I_{\mathfrak{H}} \; > -\tan^2(\alpha/2)\; I_{\mathfrak{H}} \; , \; t \geq 0 \; .$$

Now it is useful to define also the following subset of $C(\alpha)$:

$$\begin{aligned} Q(\alpha) &:= \{z \in C(\alpha) : |z\sin\alpha - \cos\alpha| \leq 1\} = \\ &= \{z \in \mathbb{C} : 2|\mathrm{Im}\,z| \leq (1-|z|^2)\tan\alpha\} \cap \{z \in \mathbb{C} : -2\mathrm{Re}\,z \leq (1-|z|^2)\tan\alpha\} \; . \end{aligned} \tag{3.9}$$

So, $Q(\alpha)$ is the intersection of *three* closed disks, cf. (2.3).

Proposition 3.5. *The set $Q(\alpha)$ is a multiplicative semigroup on $\mathbb{C}$.*

Proof. First we note that $Q(\alpha)$ contains the set $C_+(\alpha) := \{z \in C(\alpha) : \mathrm{Re}\,z \geq 0\}$. Moreover, since $\mathrm{Re}\,iz = -\mathrm{Im}\,z$, by (3.9) we obtain also that the set

$$B(\alpha) := \{z \in C(\alpha) : iz \in C(\alpha)\} \; ,$$

has a non-empty intersection with $Q(\alpha)$, such that

$$B(\alpha) \cap Q(\alpha) = B(\alpha) \;\text{ and }\; Q(\alpha) \setminus B(\alpha) \subset C_+(\alpha) \; . \tag{3.10}$$

In [4] it is shown that the set $B(\alpha)$ is the *ideal* of the multiplicative semigroup $C(\alpha)$, i.e., $z\xi \in B(\alpha)$ for all $z \in B(\alpha)$ and all $\xi \in C(\alpha)$. Let $z, \xi \in Q(\alpha) \setminus B(\alpha)$. Since $B(\alpha)$ has the properties (3.10), then

$$\arg z, \arg\xi \in (-\pi/4, \pi/4).$$

Finally, since $z\xi \in C(\alpha)$ and $\arg(z\xi) \in (-\pi/2, \pi/2)$, we obtain $z\xi \in Q(\alpha)$. □

Notice that in particular Proposition 3.5 yields:

$$z \in Q(\alpha) \Rightarrow z^n \in Q(\alpha).$$

Moreover, if $\varphi_n(z) = z^n$, then the set $\varphi_n(Q(\alpha))$ is also a multiplicative semigroup and

$$\varphi_n(Q(\alpha)) \subset Q(\alpha).$$

Since by (2.3) and (3.9) one gets that

$$C(\alpha) = Q(\alpha) \cup \{-Q(\alpha)\},$$

where $\{-Q(\alpha)\} := \{z : -z \in Q(\alpha)\}$, by (3.1) and (3.10) we have

$$\Omega(\alpha) = f(Q(\alpha))$$

when $f(z) = z^2$. Thus, we obtain:

$$\Omega(\alpha) \subset Q(\alpha). \tag{3.11}$$

Remark 3.6. *Another way to check (3.11) is the following argument. Let $\xi = z^2$, where $z \in C(\alpha)$. Then $\xi \in \Omega(\alpha)$, see (3.1). Since* $\mathrm{Re}\,\xi = (\mathrm{Re}\,z)^2 - (\mathrm{Im}\,z)^2$ *and* $(\mathrm{Im}\,z)^2 \leq |\mathrm{Im}\,z|$, *we also get*

$$\begin{aligned} -2\mathrm{Re}\,\xi = -2(\mathrm{Re}\,z)^2 + 2(\mathrm{Im}\,z)^2 \leq 2|\mathrm{Im}\,z| \leq \tan\alpha(1-|z|^2) = \\ = \tan\alpha(1-|\xi|) \leq \tan\alpha(1-|\xi|^2) \; , \end{aligned}$$

which means by (3.9) that $\Omega(\alpha) \subset Q(\alpha)$.

Remark 3.7. *Let D_α be the set (1.4) introduced in* [8]*. From definition (3.9) one deduces that $Q(\alpha) \subset D_\alpha$, as well as that D_α is not a subset of $C(\alpha)$, i.e., $D_\alpha \cap C(\alpha) \neq D_\alpha$. In addition, by virtue of (2.7) and (3.2) we obtain: $L(\alpha) \subset \Omega(\alpha)$, where $L(\alpha)$ is defined by (2.7). So, for all $\alpha \in [0, \pi/2)$ and $F(\lambda) = (I_{\mathfrak{H}} + \lambda S)^{-1}$, $\lambda > 0$ we have the following inclusions* (*see Figure 1*):

$$\begin{aligned} &L(\alpha) \subset \Omega(\alpha) \subset Q(\alpha) \subset D_\alpha \ , \\ &W(F(\lambda)) \subseteq L(\alpha) \ , \\ &W(F^n(\lambda)) \subseteq C(\alpha) \quad \textit{and} \quad W(F^{2n}(\lambda)) \subseteq \Omega(\alpha) \quad \textit{for all} \quad n \in \mathbb{N} \ , \end{aligned}$$

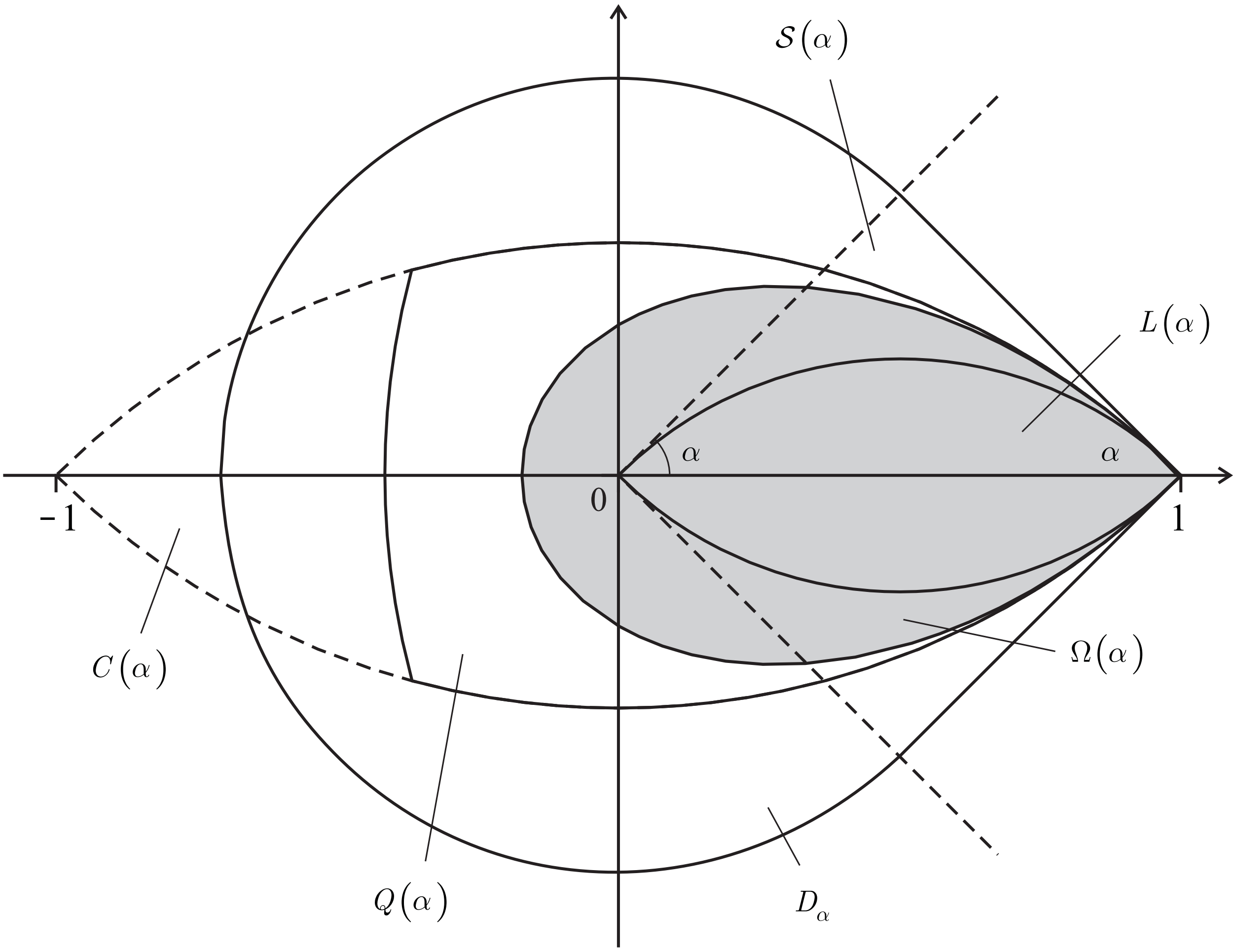

Figure 1

4. Operator-norm convergence of the Euler formula

4.1. **A brief history.** Theorem 5.1 in [8] states that

$$\left\| (I_{\mathfrak{H}} + tS/n)^{-n} - \exp(-tS) \right\| = O\left(\ln n/n\right) \tag{4.1}$$

for m-α-sectorial boundedly invertible operator S in $\mathfrak{H}$ and for all $t \in S_{\pi/2-\alpha}$. The proof of (4.1) is essentially based on two estimates for a quasi-sectorial contraction C, i.e. $W(C) \subseteq D_\alpha$, also established in [8]. The first is

$$\left\| C^n - C^{n+1} \right\| \leq \frac{K}{n+1} \tag{4.2}$$

for all $n \in \mathbb{N}$ and for some $K > 0$ depending on α, and the second one is

$$\| C^n - \exp\left(n(C - I_{\mathfrak{H}})\right) \| = O\left(1/n^{1/3}\right) \ . \tag{4.3}$$

Then application of (4.2) and (4.3) for the particular case of the operator $C = (I_{\mathfrak{H}}+tS/n)^{-1}$ leads to (4.1).

Later the estimate (4.1) was improved in [16] to

$$\|C^n - \exp\left(n(C - I_{\mathfrak{H}})\right)\| = O\left((\ln n/n)^{1/2}\right), \tag{4.4}$$

and in [6] to

$$\|C^n - \exp\left(n(C - I_{\mathfrak{H}})\right)\| = O\left(1/n^{1/2}\right). \tag{4.5}$$

In [16] the following bound is proved for the operator-norm convergence rate:

$$\left\|(I_{\mathfrak{H}} + tS/n)^{-n} - \exp(-tS)\right\| \le c/n, \tag{4.6}$$

where c is a constant depending on the operator S. The bounds (4.3), (4.4), (4.5), and (4.6) are obtained in [16] and [6] by means of the probability theory methods (the *Poisson distribution* and the *Central Limit Theorem*) that improves the estimate (4.3).

Here, using the results obtained in [11], [5], [10], and [9], and generalizations of some von Neumann inequality [15], we prove the theorem, which makes more explicit the right-hand side of the estimate (4.6).

Theorem 4.1. *Let S be an m-α-sectorial operator in a Hilbert space $\mathfrak{H}$. Then*

$$\left\|(I_{\mathfrak{H}} + tS/n)^{-n} - \exp(-tS)\right\| \le K(\alpha)/(n\cos^2\alpha)\ ,\ t \ge 0\ ,$$

where

$$(\pi \sin\alpha)/2\alpha \le K(\alpha) \le \min\left(2 + 2/\sqrt{3}\ ,\ (\pi - \alpha)/\alpha\right)\ .$$

4.2. **The von Neumann inequality and its generalizations.** The spectral sets theory was introduced by von Neumann [15] in order to extend the functional calculus to the case of non-normal operators in Hilbert spaces.

Definition 4.2. [15]. *A set $\sigma \subset \mathbb{C}$ is a spectral set of the operator A in a Hilbert space $\mathfrak{H}$ if it is closed and if for any bounded rational function $u(z)$ on σ one has*

$$||u(A)|| \le \sup_{z\in\sigma} |u(z)|.$$

Proposition 4.3. [15]. *A necessary and sufficient condition for one of the domains*

$$|z - a| \le r,\ |z - a| \ge r,\ \mathrm{Re}\,(az) \ge b$$

to be a spectral set of A in $\mathfrak{H}$ is that

$$||A - aI_{\mathfrak{H}}|| \le r,\ ||(A - aI_{\mathfrak{H}})^{-1}|| \le \frac{1}{r},\ \mathrm{Re}\,(aT) \ge b.$$

Definition 4.4. *Let $\mathcal{D}$ be an open convex subset of the complex plane ($\mathcal{D} \ne \emptyset,\ \mathcal{D} \ne \mathbb{C}$) and let A be a linear operator in a Hilbert space $\mathcal{H}$ with $W(A) \subset \overline{\mathcal{D}}$. The set $\mathcal{D}$ is called K-spectral set for the operator A, if*

$$||u(A)|| \le K \sup_{z\in\mathcal{D}} |u(z)|$$

for all rational functions $u(z)$ without pole in the spectrum of A.

Let $\mathcal{D}_k := \{|z - a_k| < r_k\}$, $k = 1, 2$ be two disks such that $\mathcal{D}_k \cap \mathcal{D}_2 = \{\xi_1, \xi_2\}$. Let

$$\mathfrak{L} := \mathcal{D}_1 \cap \mathcal{D}_2.$$

The set $\mathfrak{L}$ is said to be a *convex lens-shaped* domain [5]. Denote by $2\alpha \in (0, \pi)$ the angle of the lens L at the vertices. The operator $A \in \mathcal{L}(\mathfrak{H})$ is called *of the lenticular $\mathfrak{L}$-type* [5] if

$$||A - a_1 I_{\mathfrak{H}}|| \le r_1 \quad \text{and} \quad ||A - a_2 I_{\mathfrak{H}}|| \le r_2.$$

The next Proposition is established in [5].

Proposition 4.5. *Let $\mathfrak{L}$ be a convex lens-shaped domain of the complex plane with angle 2α. There exists a best positive constant $K(\alpha)$ such that the inequality*

$$||p(A)|| \leq K(\alpha) \sup_{z \in \mathfrak{L}} |p(z)|, \tag{4.7}$$

holds for all polynomials $p(z)$, for all operators $A \in \mathcal{L}(\mathfrak{H})$ of $\mathfrak{L}$-type, and for all Hilbert spaces $\mathfrak{H}$. The constant $K(\alpha)$ depends only on α and satisfies the inequality

$$\frac{\pi}{2\alpha} \sin\alpha \leq K(\alpha) \leq \min\left(2 + \frac{2}{\sqrt{3}}, \ \frac{\pi - \alpha}{\alpha}\right).$$

Notice that by the Mergelyan Theorem [17], the inequality (4.7) remains valid if p is holomorphic in $\mathfrak{L}$ and continuous in $\overline{\mathfrak{L}}$.

4.3. **Proof of the Theorem 4.1.**

Proof. Let

$$\mathcal{D}_1 = \left\{ z \in \mathbb{C} : \left| z - \frac{1}{2} - \frac{i}{2} \cot\alpha \right| < \frac{1}{2\sin\alpha} \right\},$$
$$\mathcal{D}_2 = \left\{ z \in \mathbb{C} : \left| z - \frac{1}{2} + \frac{i}{2} \cot\alpha \right| < \frac{1}{2\sin\alpha} \right\}.$$

Then (see (2.7)) $\mathfrak{L} = L(\alpha) \setminus \partial L(\alpha)$, i.e. interior of the set $L(\alpha)$.

Fix $t \geq 0$ and let

$$C = F(t) := (I_{\mathfrak{H}} + tS)^{-1} \ .$$

Then from (2.6) it follows that the operator C is of $\mathfrak{L}$-type.

Since

$$F(t/n) = F(t)\{(n-1)F(t)/n + I_{\mathfrak{H}}/n\}^{-1} \ , \ n \in \mathbb{N} \ ,$$

we get also that

$$F^n(t/n) = C^n\{(n-1)C/n + I_{\mathfrak{H}}/n\}^{-n} \ .$$

Put

$$h_n(z) := \exp(1 - 1/z) - z^n\{(n-1)z/n + 1/n\}^{-n} \ , \ z \in \mathfrak{L} \ .$$

Because $\operatorname{Re} z > 0$ for all $z \in \mathfrak{L}$, the function $h_n(z)$ is holomorphic in $\mathfrak{L}$ and continuous in $\overline{\mathfrak{L}} = L(\alpha)$. Moreover, since

$$\exp(-tS) = \exp(I_{\mathfrak{H}} - C^{-1}) \ ,$$

we obtain that

$$\exp(-tS) - (I_{\mathfrak{H}} + tS/n)^{-n} = h_n(C) \ .$$

The fractional-linear conformal transformation $w \mapsto z = 1/(1+w)$ maps the sector $\mathcal{S}(\alpha)$ onto $L(\alpha)$. So, let

$$g_n(w) := h_n((1+w)^{-1}) = \exp(-w) - (1 + w/n)^{-n} \ , \ w \in (\mathcal{S}(\alpha) \setminus \partial\mathcal{S}(\alpha)) \ .$$

Then clearly:

$$\sup_{z \in \mathfrak{L}} |h_n(z)| = \sup_{w \in (\mathcal{S}(\alpha) \setminus \partial\mathcal{S}(\alpha))} |g_n(w)| = \sup_{w \in \partial\mathcal{S}(\alpha)} |g_n(w)| \ .$$

Since

$$\partial\mathcal{S}(\alpha) = \{x \exp(-i\alpha), \ x \in \mathbb{R}_+\} \cup \{x \exp(i\alpha), \ x \in \mathbb{R}_+\} \ ,$$

we have to estimate the value of

$$\sup_{x \in \mathbb{R}_+} |g_n(x \exp(\pm i\alpha)| \ .$$

To this end we use the representation:

$$g_n(x\exp(i\alpha)) = -\int\limits_0^x \frac{d}{ds}\left((1+se^{i\alpha}/n)^{-n}\,\exp(-(x-s)e^{i\alpha})\right) ds \ ,$$

where

$$\frac{d}{ds}\left((1+se^{i\alpha}/n)^{-n}\,\exp(-(x-s)e^{i\alpha})\right) = \frac{se^{i\alpha}}{n}\,(1+se^{i\alpha}/n)^{-(n+1)}\,\exp(-(x-s)e^{i\alpha}) \ .$$

By elementary inequalities:

$$\begin{aligned}|1+se^{i\alpha}/n|^{n+1} &= (1+s^2/n^2+2s\cos\alpha/n)^{(n+1)/2} \geq (1+(s\cos\alpha)/n)^{n+1} \geq \\ &\leq 1+((n+1)s\cos\alpha)/n \geq 1+s\cos\alpha,\end{aligned}$$

we obtain

$$s\ |1+(s\cos\alpha)/n|^{-(n+1)} \leq s/(1+s\cos\alpha) \leq 1/\cos\alpha \ .$$

Therefore, we obtain as an upper bound:

$$|g_n(x\exp(i\alpha))| \leq \int\limits_0^x \frac{\exp(-(x-s)\cos\alpha)}{n\cos\alpha} ds \leq \frac{1}{n\cos^2\alpha}(1-\exp(-x\cos\alpha)) \leq \frac{1}{n\cos^2\alpha}.$$

Similarly one obtains the estimate:

$$|g_n(x\exp(-i\alpha))| \leq \frac{1}{n\cos^2\alpha}.$$

Thus,

$$\sup_{x\in\mathbb{R}_+} |g_n(x\exp(\pm i\alpha)| \leq \frac{1}{n\cos^2\alpha}.$$

Now by Proposition4.5 we get

$$\left\|\exp(-tS)-(I_{\mathfrak{H}}+tS/n)^{-n}\right\| = ||h_n(C)|| \leq \frac{K(\alpha)}{n\cos^2\alpha} \ , \ t\geq 0 \ , \tag{4.8}$$

which completes the proof. □

5. Conclusion

Now several remarks are in order:

(a) Theorem 2.1 from [8] states that for quasi-sectorial contractions one has:

$$W(\exp(-tS)) \subseteq D_\alpha, \ \alpha\in[0,\pi/2), \tag{5.1}$$

Here S stands for m-α-sectorial generator of contraction. As it is indicated in [18], the proof in [8] has a flaw. In [18] it was corrected but only for the range $\alpha\in[0,\pi/4]$. Because

$$\Omega(\alpha)\subset Q(\alpha)\subset D_\alpha,$$

our main Theorem 3.4 shows that the original claim (5.1) in [8] is indeed correct for all $\alpha\in[0,\pi/2)$.

(b) Since in Theorem 3.4 we proved in fact that $W(\exp(-tS))\subseteq\Omega(\alpha)$, this means we give in the present papers more precise localization of $W(\exp(-tS))$ than it was claimed in [8].

(c) The inclusion (5.1) plays an important rôle for the operator-norm error estimates of the norm convergence in the Euler formula (1.2), see the recent paper [18], the references quoted there and in particular [7]. In the present paper we give a new proof of the operator-norm convergent Euler formula for the optimal error estimate with explicit indication of its $\alpha-$dependence (4.8) for m-α-sectorial generators.

Acknowledgements

V.Z. is thankful to Pavel Bleher for useful remarks related to the multiplicative semigroups on the complex plane.

References

[1] Yu.M. Arlinskiĭ. A class of contractions in Hilbert space. Ukrain. Matematicheskii Zhurnal, 39 (1987), No.6, 691-696(Russian). English translation in: Ukrainian Mathematical Journal, 39 (1987), No.6, 560–564.
[2] Yu.M. Arlinskiĭ. Characteristic functions of operators of the class $C(\alpha)$. Izv.Vyssh. Uchebn.Zaved.Mat. 1991, N2, 13-21(Russian).
[3] Yu.M. Arlinskiĭ. Closed sectorial forms and one-parameter contraction semigroups, Matematicheskie Zametki, 61 (1997), No.5, 643-654 (Russian). English translation in: Mathematical Notes, 61 (1997), No. 5, 537–546.
[4] Yu.M. Arlinskiĭ. On some semigroups on the complex plane. Semigroup Forum, 70 (2005), No.3, 329–346.
[5] B. Beckermann and M. Crouzeix, A lenticular version of a von Neumann inequality, Archiv der Mathematik, 86 (2006), 352–355.
[6] V. Bentkus and V. Paulauskas, Optimal error estimates in operator-norm approximations of semigroups, Lett. Math. Phys. 68 (2004) 131–138.
[7] V. Cachia. Euler's exponential formula for semigroups. Semigroup Forum, 68 (2004), No.1, 1–24.
[8] V. Cachia and V.A. Zagrebnov, Operator-norm approximation of semigroups by quasi-sectorial contractions, Journ. Funct. Anal., 180 (2001), 176-194.
[9] M. Crouzeix, Numerical range and functional calculus in Hilbert space. Journal of Functional Analysis, vol. 244 (2007), 668-690.
[10] M. Crouzeix and B. Delyon, Some estimates for analytic functions of the strip or a sectorial operators, Archiv der Mathematik, 81, (2003), 559–566.
[11] B. Delyon and F. Delyon, Generalization of von Neumann's spectral sets and integral representation of operators, Bull. Soc. Math. Fr., 127 (1999), No. 1, 25–42.
[12] P.R. Chernoff, *Product formulas, nonlinear semigroups, and addition of unbounded operators.* Memoirs of the American Mathematical Society, No. 140. American Mathematical Society, Providence, R. I., 1974.
[13] T. Kato Some mapping theorem for the numerical range, Proc.Japan Acad. 41 (1966), 652–655.
[14] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, Heidelberg, 1995.
[15] J. von Neumann, Eine spectraltheorie für allgemeine operatore eines unitären raumes, Math. Nachrichten, 4 (1951), 251–281.
[16] V. Paulauskas, On operator-norm approximation of some semigroups by quasi-sectorial operators, J. Funct. Anal. 207 (2004) 58–67.
[17] W. Rudin, *Real and complex analysis*, McGraw-Hill Book Co., New York, 1987.
[18] V.A. Zagrebnov. Quasi-Sectorial Contractions, Journ. Funct. Anal., 254 (2008), 2503–2511.
[19] V.A. Zagrebnov, *Topics in the theory of Gibbs semigroups.* Leuven Notes in Mathematical and Theoretical Physics. Series A: Mathematical Physics, 10. Leuven University Press, Leuven, 2003.

Department of Mathematical Analysis, East Ukrainian National University, Kvartal Molodyozhny 20-A, Lugansk 91034, Ukraine
E-mail address: yury_arlinskii@yahoo.com; yma@snu.edu.ua

Université de la Méditerranée and Centre de Physique Théorique - UMR 6207, Luminy - Case 907, Marseille 13288, Cedex 9, France
E-mail address: zagrebnov@cpt.univ-mrs.fr